The radical of a vertex operator algebra

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1. Introduction

Suppose that V is a vertex operator algebra with canonical $\mathbb{Z}$-grading

$$V = \coprod_{n\in\mathbb{Z}} V_n.$$

Each $v \in V$ has a vertex operator $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ attached to it, where $v_n \in \mathrm{End}V$. For the conformal vector ω we write $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. If v is *homogeneous of weight* k, that is $v \in V_k$, then one knows that

$$v_n : V_m \to V_{m+k-n-1}$$

and in particular the *zero mode* $o(v) = v_{\mathrm{wt}v-1}$ induces a linear operator on each V_m. We extend the "o" notation linearly to V, so that in general $o(v)$ is the sum of the zero modes of the homogeneous components of v. Then we define the *radical* of V to be

$$J(V) = \{v \in V | o(v) = 0\} \tag{1.1}.$$

The problem arose in some work of the first three authors in [DLiM] and in work of the fourth author in [M] of describing $J(V)$ precisely. We will essentially solve this problem in the present paper in an important special case, namely that V is a vertex operator algebra of *CFT type.* This means that the $\mathbb{Z}$-grading on V has the shape

$$V = \coprod_{n=0}^{\infty} V_n \tag{1.2}$$

and moreover that $V_0 = \mathbb{C}\mathbf{1}$ is spanned by the vacuum vector $\mathbf{1}$.

V is said to be a vertex operator algebra of *hermitian* CFT type if, in addition, V has the structure of a Hilbert space and further we have an involution $v \mapsto \overline{v}$ on V such that

$$(\overline{v})_n{}^{\dagger} = \left(e^{L(1)}v\right)_{-n},$$

and $\overline{\omega} = \omega$.

In [M], the concept of a *deterministic* conformal field theory (a vertex operator algebra of hermitian CFT type in our current notation) was introduced as one for which $o(v) = 0$ and $L(1)v = 0$ imply that $v \in V_1$. Equivalently, the definition may be restated in terms of the modes $p(v) \equiv v_{\mathrm{wt}v-2}$, *i.e.* if $L(1)v = 0$ and $p(v) = 0$ then $v \in V_0$. The motivation is that if we have a state $v \in \coprod_{n=2}^{\infty} V_n$ such that $p(v) = 0$ then $o\left(\frac{1}{L(0)-1}v\right)$ is a generator of a (possibly trivial) continuous symmetry of the conformal field theory. We show in this paper that all conformal field theories are deterministic (restricting attention only to states which are finite sums of components coming from distinct V_n's).

To describe our result, let

$$J_1(V) = J(V) \cap V_1. \tag{1.3}$$

We observe that $J(V)$ is in general *not* a $\mathbb{Z}$-graded subspace of V, which accounts for some of the difficulty that its study offers, nevertheless the radical elements of weight one will turn out to play a special rôle. We will prove

THEOREM 1. *Suppose that V is a vertex operator algebra of CFT type. Then*

$$J(V) = J_1(V) + (L(0) + L(-1))V.$$

Related to Theorem 1 are the next two theorems.

THEOREM 2. *Suppose that V is a vertex operator algebra of CFT type. Let v be homogeneous of weight at least one. Then there is an integer t such that $0 \le t \le \mathrm{wt}v$ and*

$$Y(v,z) = \sum_{n<0} v_n z^{-n-1} + \sum_{n\ge t} v_n z^{-n-1} \tag{1.4}$$

where each operator v_n in (1.4) is non-zero.

We define the *degree* of v to be t and write $\deg v = t$ if v satisfies the hypotheses and conditions of Theorem 2. For general $v \in V$ define $\deg v$ to be the least of the degrees of the homogeneous components of v, and $\deg v = -1$ if $v \in V_0$. For $d \ge 0$ set

$$V^d = \{v \in V \,|\, \deg v \ge d\}.$$

Then $V^0 = \coprod_{n\ge1} V_n$ and $V^0 \supset V^1 \supset V^2 \supset \cdots$ defines a filtration on V^0. We prove

THEOREM 3. *Suppose that V is a vertex operator algebra of CFT type. If $d \ge 1$ then*

$$V^d = L(-1)^{d-1} J_1(V) + L(-1)^d V.$$

In favorable situations we will have $J_1(V) = 0$, in which case Theorems 1 and 3 take a simpler form. Although we will say something about this situation below, we do not yet have a complete description of those vertex operator algebras of CFT type for which $J_1(V)$ is not zero. We note here only that the vertex operator algebra associated with the Heisenberg algebra has $J_1(V) \neq 0$; indeed $J_1(V) = V_1$ in this case.

We refer the reader to [FHL], [FLM], [DGM] and [DHL] for background definitions and elementary results about vertex operator algebras.

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2. Quasi-primary elements

Recall that $v \in V$ is called *quasi-primary* in case $L(1)v = 0$. It is convenient to introduce the term *semi-primary* for an element $v \in V$ if v is either quasi-primary or of weight 1. Note that v is quasi-primary or semi-primary if, and only if, each homogeneous component of v has the same property. Also note that, for a vertex operator algebra of hermitian CFT type, all weight one states are quasi-primary and so the terms quasi- and semi-primary are synonomous.

In this section we prove the following results under the assumption that V is a vertex operator algebra of CFT type.

THEOREM 2.1. *If $v \in V$ is homogeneous and satisfies $v_{\mathrm{wt} v} = 0$, then $v \in V_0$.*

THEOREM 2.2. *Suppose that $v \in V$ is semi-primary and satisfies $o(v) = 0$. Then $v \in V_1$.*

THEOREM 2.3. *Suppose that $v \in V$ is quasi-primary and homogeneous of weight at least 2. If $v_0 = 0$ then $v = 0$.*

The following is well-known.

LEMMA 2.4. *The following are equivalent for a homogeneous element $v \in V$:*
(a) $v \in V_0$.
(b) $[L(-1), v_n] = 0$ for all $n \in \mathbb{Z}$.

PROOF:—
This follows from the identity

$$[L(-1), Y(v, z)] = Y(L(-1)v, z) = \frac{d}{dz} Y(v, z) \tag{2.1}$$

together with the fact that $L(-1)$ is injective on V_n for $n \neq 0$ (cf. Corollary 2.4 of [DLM]). □

Turning to the proof of Theorem 2.1, the component version of (2.1) is

$$[L(-1), v_n] = (L(-1)v)_n = -nv_{n-1}. \tag{2.2}$$

So as $v_{\mathrm{wt}v} = 0$ then we see inductively that

$$v_n = 0 \tag{2.3}$$

for $0 \leq n \leq \mathrm{wt}v$.

Now in general one has [FLM]

$$[a_m, b_n] = \sum_{t \geq 0} \binom{m}{t} (a_t b)_{m+n-t}. \tag{2.4}$$

Taking $b_n = L(-1) = \omega_0$ in (2.4) and taking (2.3) into account, we see that

$$[v_m, L(-1)] = \sum_{t > \mathrm{wt}v} \binom{m}{t} (v_t \omega)_{m-t}. \tag{2.5}$$

Now $\mathrm{wt}v_t\omega = \mathrm{wt}v - t + 1$, so $v_t\omega = 0$ if $t > \mathrm{wt}v + 1$. So (2.5) reduces to

$$[v_m, L(-1)] = \binom{m}{\mathrm{wt}v + 1} (v_{\mathrm{wt}v+1}\omega)_{m-\mathrm{wt}v-1}. \tag{2.6}$$

However, $\mathrm{wt}v_{\mathrm{wt}v+1}\omega \in V_0$, so $(v_{\mathrm{wt}v+1}\omega)_{m-\mathrm{wt}v-1} = 0$ unless $m - \mathrm{wt}v - 1 = -1$, that is $m = \mathrm{wt}v$. In this case $\binom{m}{\mathrm{wt}v+1} = 0$. So (2.6) yields

$$[v_m, L(-1)] = 0 \tag{2.7}$$

for all $m \in \mathbb{Z}$.

Theorem 2.1 is now a consequence of (2.7) and Lemma 2.4.

We next present the proof of Theorem 2.2. Denote the homogeneous components of v by v^i, that is

$$v = \sum_{i \geq 0} v^i, \quad v^i \in V_i. \tag{2.8}$$

Note that we do not necessarily have $o(v^i) = 0$, although each v^i is certainly semi-primary.

LEMMA 2.5. *The following hold:*

(i) If u is a homogeneous and quasi-primary then

$$[L(1), u_t] = 2(\mathrm{wt}u - t/2 - 1)u_{t+1}. \tag{2.9}$$

(ii) If u has weight 1 then

$$[L(1), u_0] = 0. \tag{2.10}$$

PROOF:—

Use (2.2) and (2.4) with $a_m = L(1) = \omega_2$ to see that

$$[L(1), u_t] = (2\mathrm{wt}u - t - 2)u_{t+1} + (L(1)u)_t.$$

So if u is quasi-primary then (2.9) holds. If $\mathrm{wt}u = 1$ and $t = 0$ then $[L(1), u_0] = (L(1)u)_0$. But $L(1)u \in V_0 = \mathbb{C}\mathbf{1}$, so that $(L(1)u)_0 = 0$. $\square$

Turning to the proof of Theorem 2.2, since $o(v) = 0$ we see from (2.8)-(2.10) that

$$0 = [L(1), o(v)] = \sum_{i\geq 0}[L(1), o(v^i)] = \sum_{i\geq 2} 2(\mathrm{wt}v^i - (\mathrm{wt}v^i - 1)/2 - 1)v^i_{\mathrm{wt}v^i}.$$

So

$$\sum_{i\geq 2}(i-1)v^i_i = 0. \tag{2.11}$$

We now apply the operator $[L(1), [L(-1), *]]$ to (2.11). We have

$$[L(1), [L(-1), u_t]] = [L(1), -tu_{t-1}] = -t(2\mathrm{wt}u - t - 1)u_t$$

by (2.9). So (2.11) implies that

$$\sum_{i\geq 2}(i-1)i(i-1)v^i_i = 0.$$

Continuing in this fashion, we get

$$\sum_{i\geq 2}(i-1)^k i^{k-1} v^i_i = 0$$

for all $k \geq 1$. It follows that each $v^i_i = 0$ for all $i \geq 2$, and therefore $v^i = 0$ for $i \geq 2$ by Theorem 2.1. Hence

$$v = v^0 + v^1.$$

To finish the theorem we need to also show that $v^0 = 0$. In fact we have $o(v) = v^0_{-1} + v^1_0 = 0$, so that $0 = v^0_{-1}\mathbf{1} + v^1_0\mathbf{1}$. But $v^1_0\mathbf{1} = 0$, so $v^0 = v^0_{-1}\mathbf{1} = 0$, and the proof of Theorem 2.2 is complete.

To prove Theorem 2.3, let v be as in the statement of the theorem. Then $v_0 = 0$, and we can apply (2.9) to see inductively that $v_n = 0$ for $0 \leq n \leq 2(\mathrm{wt}v - 1)$. As $\mathrm{wt}v \geq 2$ then in particular we get $v_{\mathrm{wt}v} = 0$ and so we can apply Theorem 2.1 to complete the proof.

3. The shape of a vertex operator

We prove Theorems 2 and 3 in this section. Let $v \in V$ be a non-zero homogeneous vector of weight at least one and with vertex operator

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}.$$

We have $Y(v,z)\mathbf{1} = e^{zL(-1)}v$, in particular if $n < 0$ then $v_n\mathbf{1} = \frac{L(-1)^{-n-1}}{(-n-1)!}v$. As $L(-1)$ is injective on V_n for $n \neq 0$ we deduce that $v_n\mathbf{1} \neq 0$, so that $v_n \neq 0$ for $n < 0$.

Next, if $v_n = 0$ with $n \geq 0$ then (2.2) yields $0 = [L(-1), v_n] = -nv_{n-1}$. This then shows that $v_m = 0$ for $0 \leq m \leq n$. By Theorem 2.1 we find that $v_n \neq 0$ for $n \geq \mathrm{wt}v$, and now Theorem 2 follows immediately.

Now we prove Theorem 3. We start with

LEMMA 3.1. *If $v \in V$ is homogeneous of degree t, then $L(-1)^k v$ has degree $k+t$ for all $k \geq 0$.*

PROOF:—
We may take $k = 1$. Now $Y(v,z)$ has the shape (1.4), whence

$$Y(L(-1)v,z) = \frac{d}{dz}Y(v,z) = -\sum_{n<0}(n+1)v_n z^{-n-2} - \sum_{n\geq t}(n+1)v_n z^{-n-2}$$

i.e.,

$$Y(L(-1)v,z) = -\sum_{n<0} nv_{n-1}z^{-n-1} - \sum_{n\geq t+1} nv_{n-1}z^{-n-1}. \tag{3.1}$$

Since each v_n is (3.1) is non-zero we see that $\deg L(-1)v = t+1$ as required. □

COROLLARY 3.2. *If $d \geq 1$ then*

$$L(-1)^{d-1}J_1(V) + L(-1)^d V \subset V^d \subset \bigoplus_{n\geq d} V_n.$$

PROOF:—
$L(-1)^{d-1}J_1(V) + L(-1)^d V \subset V^d$ follows from Lemma 3.1. The containment $V^d \subset \oplus_{n\geq d}V_n$ reflects the fact that $\deg v \leq \mathrm{wt}v$. □

We also need

LEMMA 3.3. *If $n \neq 1$ there is a direct sum decomposition*

$$V_n = \ker(L(1): V_n \to V_{n-1}) \bigoplus \mathrm{im}(L(-1): V_{n-1} \to V_n).$$

PROOF:—
See, for example, Proposition 3.4 of [DLM]. □

To Prove Theorem 3, suppose that there is v of degree d with $v \notin L(-1)^{d-1}J_1(V) + L(-1)^d V$. By Lemma 3.3 we can choose v to be homogeneous and such that it has an expression of the form

$$v = \sum_{n=0}^{d-1} L(-1)^n u^n$$

with each u^n homogeneous and semi-primary. Moreover we can take $\mathrm{wt}v = \mathrm{wt}(L(-1)^n u^n)$, i.e., $\mathrm{wt}v = n + \mathrm{wt}u^n$ for $0 \leq d-1$.

Now if $\mathrm{wt}u^n = 1$ then $\mathrm{wt}v = n+1$, so as $n \leq d-1$ then $\mathrm{wt}v \leq d$. So in fact $\mathrm{wt}v = d$ and $n = d-1$ in this case. So if $n \leq d-2$ then u^n is quasi-primary of weight at least 2, so that $\deg(L(-1)^n u^n) = n$ if $u^n \neq 0$ by Lemma 3.1 and Theorem 2.3. Since v has degree $d \geq n+1$ and $\deg(L(-1)^{d-1}u^{d-1}) \geq d-1$ we see that $L(-1)^n u^n = 0$ for $n \leq d-2$.

So now $v = L(-1)^{d-1}u^{d-1}$ with $\mathrm{wt}u^{d-1} = 1$, forcing $\deg u^{d-1} = 1$ by Lemma 3.1. So $u^{d-1} \in J_1(V)$, and the theorem follows. $\square$

4. Theorem 1

First we prove

Lemma 4.1. $J_1(V) + (L(0) + L(-1))V \subset J(V)$.

Proof:—

It is enough to show that $(L(0) + L(-1))v$ lies in $J(V)$ for homogeneous $v \in V$. But we have $L(0)v = (\mathrm{wt}v)v$ and $(L(-1)v)_n = -nv_{n-1}$, so taking $n = \mathrm{wt}(L(-1)v) - 1 = \mathrm{wt}v$ shows that $o(L(0)v + L(-1)v) = 0$ as required. $\square$

To begin the proof of Theorem 1, pick $v \in J(V)$. As before we can write

$$v = \sum_{n=0}^{m} L(-1)^n u^n \tag{4.1}$$

where each u^n is semi-primary and where $u^m \neq 0$. We prove by induction on m that v lies in $J_1(V) + (L(0) + L(-1))V$.

Suppose first that $m = 0$. Then $v = u^m$ is semi-primary, whence the condition $o(v) = 0$ forces $v \in V_1$ by Theorem 2.2. So in fact $v \in J_1(V)$ in this case.

In general, set $x = L(-1)^{m-1}u^m$ and $y = \sum_{n=0}^{m-1} L(-1)^n u^n$. Thus $v = L(-1)x + y$. Now from Lemma 4.1 we have $(L(0) + L(-1))x \in J(V)$, that is

$$0 = o(v) = o(L(-1)x) + o(y) = -o(L(0)x) + o(y) = o(y - L(0)x).$$

We easily check that $L(0)x = (m-1)L(-1)^{m-1}u^m + L(-1)^{m-1}L(0)u^m$ so that

$$y - L(0)x = \sum_{n=0}^{m-2} L(-1)^n u^n + L(-1)^{m-1}((m-1)u^m + L(0)u^m + u^{m-1})$$

lies in $J(V)$. Since $L(0)u^m$ is semi-primary, we conclude by induction that $y - L(0)x$ lies in $J_1(V) + (L(0) + L(-1))V$. But then the same is true of $v = y - L(0)x + (L(0) + L(-1))x$. This completes the proof of the theorem.

We make some remarks about Heisenberg vertex operator algebras. These are constructed from a finite-dimensional abelian Lie algebra H equipped with a non-degenerate, symmetric, bilinear form $\langle , \rangle$. One then forms the $\mathbb{Z}$-graded affine Lie algebra

$$L = H \bigotimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

where $[u \otimes t^m, v \otimes t^n] = \langle u, v\rangle m\delta_{m+n,0}c$ and $[c, L] = 0$. The corresponding vertex operator algebra has underlying Fock space

$$M(1) = S(\oplus_{n<0} H \otimes t^n)$$

with the usual action of L. In particular, c acts as 1. If $h \in H$ is identified with $h \otimes t^{-1}$ we have

$$Y(h, z) = \sum_{n \in \mathbb{Z}} (h \otimes t^n) z^{-n-1}$$

so that $h_n = h \otimes t^n$. In particular, we see that $o(h) = h_0$ acts trivially on $M(1)$, that is $o(h) = 0$. So in fact $M(1)_1 = H = J_1(M(1))$ in this case.

More generally, we see that if $M(1)$ is as above and if V is any vertex operator algebra then the tensor product $M(1) \otimes V$ (cf. [FHL]) is such that $M(1) \otimes \mathbf{1} \subset J_1(M(1) \otimes V)$.

There is a converse to this observation. To explain this, suppose that V is a vertex operator algebra of CFT type. On V_1 there is a canonical symmetric bilinear form defined by

$$\langle u, v\rangle = u_1 v.$$

Proposition 4.2. *Suppose that $H \subset J_1(V)$ is a subspace such that the restriction of $\langle , \rangle$ to H is non-degenerate. Then V is a tensor product*

$$V \simeq M(1) \bigotimes W$$

for some vertex operator algebra W.

Proof:—

Given the condition on H, it generates a sub vertex operator algebra of V (with a different Virasoro element) isomorphic to $M(1)$. Now V is a $M(1)$-module, that is an L-module (L as above). As is well-known (e.g., Theorem 1.7.3 of [FLM]) this implies that there is an isomorphism of L-modules $V \simeq M(1) \otimes W$ where W is the space of highest weight vectors for the action of L on V. That is,

$$W = \{v \in V | h_n v = 0, h \in H, n \geq 0\}.$$

This is the so-called commutant of $M(1)$ (cf. Theorem 5.1 of [FZ]), and is known (loc.cit.) to be also a vertex operator algebra . The proposition follows. $\square$

We do not know if elements of $J_1(V)$ can occur in ways other than those described above, though for a vertex operator algebra of *hermitian* CFT type, the canonical symmetric bilinear form introduced above is non-degenerate (see, for example, [DGM]), and so Proposition 4.2 provides the complete picture in this case.

We may consider $O_\infty(V)$, an object closely related to the radical of V, which we define as follows:

$$O_\infty(V) = \{v \in V : o(v)_M = 0 \text{ for all modules } M\},$$

where $o(v)_M$ is the action of the zero mode of the vertex operator corresponding to v on the module M. This is to be compared to Zhu's $O(V)$, which is the set of all states in V whose zero modes annihilate the states of lowest conformal weight in each module. (In fact, $O_\infty(V)$ may alternatively be defined as the intersection of objects $O_n(V)$ as described in [DLiM].) Clearly, $O_\infty(V) \subset J(V) = J_1(V) + (L(0)+L(-1))V$. Further, if $v \in J_1(V) \cap O_\infty(V)$, then, by Proposition 4.2, V splits up into a tensor product $M(1) \otimes W$. We have modules for $M(1)$ on which the zero mode of v is non-zero, and so tensoring with the adjoint module for W gives a module M for V with $o(v)_M \neq 0$. We deduce that $O_\infty(V) = (L(0)+L(-1))V$.

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